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Reality of superstring field theory action

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ABSTRACT: We determine the reality conditions on the string fields that make the action for heterotic and type II string field theories real.

KEYWORDS: String Field Theory, Superstrings and Heterotic Strings

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1 Introduction and summary

String field theory is a useful technique that allows us to systematically deal with the infrared divergences that arise in the usual world-sheet approach. For this program to be successful, one needs to construct a suitable string field theory whose Feynman diagrams reproduce the string theory amplitudes constructed from world-sheet description, to all orders in perturbation expansion. Such an action was constructed for bosonic string theories in [1, 2], and for type II and heterotic string theories more recently in [3].

Given a string field theory action, one important question is: what are the reality conditions on the fields that appear in the action? These conditions ensure that the action is real for arbitrary field configuration satisfying the reality conditions. One can carry out much of the analysis in the theory, e.g. proof of gauge invariance, derivation of Feynman rules, etc. without knowing the reality conditions. Nevertheless being able to find this condition is necessary for the consistency of the theory. For example this is necessary for determining the overall phase of the S-matrix, which in turn is needed for checking the unitarity of the S-matrix [4]. It is also necessary for determining which classical solutions are allowed. For example if we have a scalar field with potential proportional to $(\phi^2 + a^2)^2$, then the only translationally invariant solution is $\phi = 0$ if ϕ is required to take real values, whereas we can also have solutions with $\phi = \pm i a$ if ϕ is required to take imaginary values.

Reality conditions for the fields of bosonic string theory were determined in [2] and analyzed in more detail in [5]. In this paper we determine the reality conditions for the fields of superstring field theory. Our method differs slightly from that of [2]. So in section 2 we first illustrate this method by applying it to the bosonic string field theory. The result of our analysis agrees with that of [2]. In section 3 we apply this method to determine the reality condition on the fields of superstring field theory constructed in [3]. In section 4 we rewrite the reality condition as a relation between hermitian conjugate and BPZ conjugate of the string field, generalizing the result of [2]. In section 5 we briefly discuss extension

of our analysis to bosonic and superstring field theories in arbitrary background described by general world-sheet (super-)conformal field theory.

Throughout this paper we shall follow the conventions of [3]. These differ from those of [2] in certain aspects. For example the bracket $\{ \}$ used here was denoted simply by $\{ \}$ in [2] and the regions $\overline{\mathcal{R}}_{g,n}$ were called $\mathcal{V}_{g,n}$ in [2]. Our normalization condition for the correlation functions in the world-sheet theory is given in (2.37) which differs from the one used in [2] by a minus sign.

2 Reality condition in bosonic string field theory

The world-sheet theory of bosonic string theory contains 26 scalars X^μ for $0 \leq \mu \leq 25$, holomorphic ghost fields b, c , and anti-holomorphic ghost fields \bar{b}, \bar{c} . The singular parts of the operator product expansion of these fields have the form:

$$\begin{aligned} b(z)c(w) &= \frac{1}{z-w} + \cdots, & \bar{b}(\bar{z})\bar{c}(\bar{w}) &= \frac{1}{\bar{z}-\bar{w}} + \cdots, \\ \partial X^\mu(z)\partial X^\nu(w) &= -\frac{\eta^{\mu\nu}}{2(z-w)^2} + \cdots, & \bar{\partial} X^\mu(\bar{z})\bar{\partial} X^\nu(\bar{w}) &= -\frac{\eta^{\mu\nu}}{2(\bar{z}-\bar{w})^2} + \cdots, \end{aligned} \quad (2.1)$$

where we have set $\alpha' = 1$. On the complex plane, the fields have mode expansion

$$b(z) = \sum_n b_n z^{-n-2}, \quad c(z) = \sum_n c_n z^{-n+1}, \quad (2.2)$$

$$\bar{b}(\bar{z}) = \sum_n \bar{b}_n \bar{z}^{-n-2}, \quad \bar{c}(\bar{z}) = \sum_n \bar{c}_n \bar{z}^{-n+1},$$

$$i\sqrt{2}\partial X^\mu(z) = \sum_n \alpha_n^\mu z^{-n-1}, \quad i\sqrt{2}\bar{\partial} X^\mu(\bar{z}) = \sum_n \bar{\alpha}_n^\mu \bar{z}^{-n-1}, \quad \bar{\alpha}_0^\mu \equiv \alpha_0^\mu. \quad (2.3)$$

The Virasoro generators, defined through the mode expansion of the stress tensors $T(z)$ and $\bar{T}(\bar{z})$:

$$T(z) = \sum_n L_n z^{-n-2}, \quad \bar{T}(\bar{z}) = \sum_n \bar{L}_n \bar{z}^{-n-2}, \quad (2.4)$$

can be expressed in terms of $b_n, \bar{b}_n, c_n, \bar{c}_n, \alpha_n^\mu$ and $\bar{\alpha}_n^\mu$ without any explicit factor of i . We shall denote by \mathcal{H} the Hilbert space of states $|s\rangle$ in the combined CFT of the matter and ghost system satisfying

$$|s\rangle \in \mathcal{H} \quad \text{iff} \quad b_0^- |s\rangle = 0, \quad L_0^- |s\rangle = 0, \quad (2.5)$$

where

$$b_0^\pm \equiv (b_0 \pm \bar{b}_0), \quad L_0^\pm \equiv (L_0 \pm \bar{L}_0), \quad c_0^\pm \equiv \frac{1}{2}(c_0 \pm \bar{c}_0). \quad (2.6)$$

Let $\{|\varphi_r(k)\rangle\}$ be a complete set of basis states created by the action of b_{-n}, \bar{b}_{-n} for $n \geq 2$, c_{-n}, \bar{c}_{-n} for $n \geq -1$, and $i\alpha_{-n}^\mu, i\bar{\alpha}_{-n}^\mu$ for $n \geq 1$ on the state $|k\rangle = e^{ik \cdot X}(0)|0\rangle$, where $|0\rangle$ denotes the $\text{SL}(2, \mathbb{C})$ invariant vacuum. In that case the vertex operators of the states $|\varphi_r\rangle$ can be expressed as sum of products of $e^{ik \cdot X}$ and (derivatives of) $b, c, \bar{b}, \bar{c}, \partial X$ and $\bar{\partial} X$, without any explicit factor of i .

The string field $|\Psi\rangle$ is taken to be an arbitrary state in \mathcal{H} , and can be expanded as

$$|\Psi\rangle = \sum_r \int \frac{d^{26}k}{(2\pi)^{26}} \psi_r(k) |\varphi_r(k)\rangle. \quad (2.7)$$

If φ_r has ghost number n_r , then ψ_r and φ_r have grassmann parity $(-1)^{n_r}$ so that the string field is always grassmann even. The string field theory action has a kinetic term

$$\begin{aligned} S_K &= \frac{1}{2g_s^2} \langle \Psi | c_0^- Q_B | \Psi \rangle \\ &= \frac{1}{2g_s^2} \sum_{r,s} (-1)^{n_r n_s} \int \frac{d^{26}k}{(2\pi)^{26}} \frac{d^{26}k'}{(2\pi)^{26}} \psi_r(k) \psi_s(k') \langle \varphi_r(k) | c_0^- Q_B | \varphi_s(k') \rangle \end{aligned} \quad (2.8)$$

where g_s is the string coupling and Q_B is the BRST charge, constructed from the oscillators of b , c , \bar{b} , \bar{c} , T and \bar{T} without any explicit factor of i . $\langle A | B \rangle$ denotes BPZ inner product defined as

$$\langle A | B \rangle = \langle I \circ A(0) B(0) \rangle, \quad (2.9)$$

with $I(z) \equiv 1/z$ and $I \circ A$ denoting the conformal transform of A by the transformation I .

In order to construct the interaction term, we introduce a fiber bundle $\hat{\mathcal{P}}_{g,n}$ with base $\mathcal{M}_{g,n}$ — the moduli space of genus g Riemann surface with n punctures — and fiber labelled by the choice of local coordinates (up to phases) around each puncture [2]. We shall denote by $\Sigma_{g,n}$ a point in $\hat{\mathcal{P}}_{g,n}$ describing a specific Riemann surface with n -punctures and the choice of local coordinates on the punctures. The multi-string interaction vertex $\{A_1 \cdots A_n\}$ for arbitrary states $|A_1\rangle, \dots, |A_n\rangle \in \mathcal{H}$ is then defined as

$$\begin{aligned} \{A_1 \cdots A_n\} &= \sum_g (g_s)^{2g} (2\pi i)^{-(3g-3+n)} \\ &\quad \times \int_{\overline{\mathcal{R}}_{g,n}} dm^1 \wedge \cdots \wedge dm^{6g-6+2n} \left\langle b[v^{(1)}] \cdots b[v^{(6g-6+2n)}] \prod_{i=1}^n A_i \right\rangle_{\Sigma_{g,n}} \end{aligned} \quad (2.10)$$

where $\overline{\mathcal{R}}_{g,n}$ denotes part of a section of $\hat{\mathcal{P}}_{g,n}$ satisfying appropriate identities [2], $m^1, \dots, m^{6g-6+2n}$ are the coordinates on $\mathcal{M}_{g,n}$ which can also be taken to parametrize $\overline{\mathcal{R}}_{g,n}$, and $\langle \rangle_{\Sigma_{g,n}}$ denotes correlation function on the Riemann surface $\Sigma_{g,n}$, with the vertex operators of A_1, \dots, A_n inserted at the punctures using the local coordinate system associated with $\Sigma_{g,n}$. The $b[v^{(\alpha)}]$ factors are defined as follows. We can use standard procedure involving Schiffer variation [2] to associate with any tangent vector $\partial/\partial m^\alpha$ of $\overline{\mathcal{R}}_{g,n}$ a set of holomorphic vector fields $v^{(\alpha,i)}$ on $\Sigma_{g,n}$ for $i = 1, \dots, n$. $v^{(\alpha,i)}$ either vanishes or is well defined around the curve C_i encircling the i -th puncture, but may not be well defined away from C_i . Then $v^{(\alpha,i)}$ will have a Laurent expansion of the form

$$v^{(\alpha,i)}(w_i) = \sum_m v_m^{(\alpha,i)} w_i^{-m+1}, \quad (2.11)$$

where w_i denotes the local coordinate around the i -th puncture with the puncture situated at $w_i = 0$. In this case we define

$$b[v^{(\alpha)}] = \sum_i \left[\oint_{C_i} dw_i b(w_i) v^{(\alpha,i)}(w_i) + \oint_{\bar{C}_i} d\bar{w}_i \bar{b}(\bar{w}_i) \overline{v^{(\alpha,i)}(w_i)} \right], \quad (2.12)$$

where the definition of \oint includes the usual $1/2\pi i$ factors so that $\oint_{C_i} dw_i/w_i = 1$ and $\oint_{C_i} d\bar{w}_i/\bar{w}_i = 1$. If $b_n^{(i)}$ and $\bar{b}_n^{(i)}$ denote the usual oscillators of b and \bar{b} acting on the Hilbert space of the i -th external state, then this can also be expressed as

$$b[v^{(\alpha)}] = \sum_i \sum_m \left(v_m^{(\alpha,i)} b_{-m}^{(i)} + v_m^{(\alpha,i)*} \bar{b}_{-m}^{(i)} \right), \quad (2.13)$$

where $*$ denotes complex conjugation.

In terms of the curly bracket defined in (2.10), the interaction term of the string field theory action takes the form

$$S_I = \frac{1}{g_s^2} \sum_{n=1}^{\infty} \frac{1}{n!} \{\Psi^n\}, \quad (2.14)$$

where $\{\Psi^n\} \equiv \{\Psi\Psi \cdots \Psi\}$ with n insertions of Ψ inside the curly bracket. Note that the sum starts at $n = 1$. While the tree level action contains interaction terms involving cubic and higher powers of the string field, the Batalin-Vilkovisky (BV) quantum master action $S_K + S_I$ also receives higher genus contribution that includes linear and quadratic terms in the string field.

Let us denote by $\Sigma_{g,n}^*$ the Riemann surface of genus g and n punctures, obtained from $\Sigma_{g,n}$ by complex conjugation of all transition functions used to define $\Sigma_{g,n}$, and the local coordinates at the punctures. We also denote by $\bar{\mathcal{R}}_{g,n}^*$ the image of $\bar{\mathcal{R}}_{g,n}$ under this map. We shall assume that $\bar{\mathcal{R}}_{g,n}$ has been chosen such that [2]¹

$$\bar{\mathcal{R}}_{g,n}^* = \bar{\mathcal{R}}_{g,n}. \quad (2.15)$$

This means that for every $\Sigma_{g,n} \in \bar{\mathcal{R}}_{g,n}$, we have $\Sigma_{g,n}^* \in \bar{\mathcal{R}}_{g,n}$.

We are now ready to describe the reality condition on the string field $|\Psi\rangle$. If n_r is the ghost number of φ_r , then we impose the reality condition

$$\psi_r(k)^* = (-1)^{n_r(n_r+1)/2+1} \psi_r(-k), \quad (2.16)$$

where $\psi_r(k)$ are the coefficients of expansion appearing in (2.7). Therefore $\psi_r(k)^* = \psi_r(-k)$ when $n_r = 1$ or $2 \bmod 4$, and $\psi_r(k)^* = -\psi_r(-k)$ when $n_r = 0$ or $3 \bmod 4$. Our goal will be to show that once $\psi_r(k)$ satisfy (2.16), the string field theory action given by the sum of (2.8) and (2.14) takes real values.

If we define $\chi_r(k)$ via

$$\psi_r(k) = (i)^{n_r(n_r+1)/2+1} \chi_r(k), \quad (2.17)$$

then the reality condition may be written as

$$\chi_r(k)^* = \chi_r(-k). \quad (2.18)$$

¹Since at a generic point on the moduli space of Riemann surfaces the conjugation acts non-trivially, (2.15) requires that on the conjugate Riemann surface we choose the local coordinates to be complex conjugates of the original choice. On special Riemann surfaces which are invariant under conjugation, (2.15) requires that the local coordinates must either be invariant under conjugation, or we must average over two choices related by conjugation.

Alternatively we could have absorbed the phase factor on the right hand side of (2.17) into the definition of the basis states $|\varphi_r(k)\rangle$ so that $\chi_r(k)$ will be directly the coefficients appearing in the expansion of the string field in this basis. However we shall continue to work with the original choice of the basis states.

In terms of the variables $\chi_r(k)$ the bosonic string field theory action given by the sum of (2.8) and (2.14) may be written as

$$S = g_s^{-2} \sum_n \frac{1}{n!} \sum_{r_1, \dots, r_n} (-1)^{\sum_{i < j} n_{r_i} n_{r_j}} \times \int \frac{d^{26} k_1}{(2\pi)^{26}} \cdots \frac{d^{26} k_n}{(2\pi)^{26}} V_{r_1 \dots r_n}^{(n)}(k_1, \dots, k_n) \chi_{r_1}(k_1) \cdots \chi_{r_n}(k_n), \quad (2.19)$$

where the vertex $V^{(n)}$ is given by

$$\begin{aligned} V_{r_1 \dots r_n}^{(n)}(k_1, \dots, k_n) &= (i)^{\sum_{i=1}^n \{n_{r_i}(n_{r_i}+1)/2+1\}} \{\!\!\{ \varphi_{r_1}(k_1) \cdots \varphi_{r_n}(k_n) \}\!\!\}, \quad \text{for } n \neq 2, \\ V_{r_1 r_2}^{(2)}(k_1, k_2) &= (i)^{\sum_{i=1}^2 \{n_{r_i}(n_{r_i}+1)/2+1\}} [\langle \varphi_{r_1}(k_1) | c_0^- Q_B | \varphi_{r_2}(k_2) \rangle + \{\!\!\{ \varphi_{r_1}(k_1) \varphi_{r_2}(k_2) \}\!\!\}], \end{aligned} \quad (2.20)$$

with the ‘interaction terms’ $\{\!\!\{ \varphi_{r_1}(k_1) \cdots \varphi_{r_n}(k_n) \}\!\!\}$ for $n \leq 2$ receiving contributions from genus ≥ 1 Riemann surfaces. This has symmetry property:

$$V^{(n)} \rightarrow (-1)^{n_{r_i} n_{r_{i+1}}} V^{(n)} \quad \text{under } r_i \leftrightarrow r_{i+1}, \quad k_i \leftrightarrow k_{i+1}, \quad (2.21)$$

and satisfies ghost number and momentum conservation laws

$$V_{r_1 \dots r_n}^{(n)}(k_1, \dots, k_n) \propto \delta_{\sum_i n_{r_i}, 2n} \delta^{(26)}(k_1 + \cdots + k_n). \quad (2.22)$$

We shall show that

$$\{\!\!\{ \varphi_{r_1}(k_1) \cdots \varphi_{r_n}(k_n) \}\!\!\}^* = \{\!\!\{ \varphi_{r_1}(-k_1) \cdots \varphi_{r_n}(-k_n) \}\!\!\}, \quad (2.23)$$

and

$$\langle \varphi_{r_1}(k_1) | c_0^- Q_B | \varphi_{r_2}(k_2) \rangle^* = \langle \varphi_{r_1}(-k_1) | c_0^- Q_B | \varphi_{r_2}(-k_2) \rangle. \quad (2.24)$$

It follows from this that

$$V_{r_1 \dots r_n}^{(n)}(k_1, \dots, k_n)^* = (-1)^{\sum_{i=1}^n \{n_{r_i}(n_{r_i}+1)/2+1\}} V_{r_1 \dots r_n}^{(n)}(-k_1, \dots, -k_n). \quad (2.25)$$

Using (2.21) this can be written as

$$\begin{aligned} V_{r_1 \dots r_n}^{(n)}(k_1, \dots, k_n)^* &= (-1)^{\sum_i \{n_{r_i}(n_{r_i}+1)/2+1\}} (-1)^{\sum_{i < j} n_{r_i} n_{r_j}} V_{r_n \dots r_1}^{(n)}(-k_n, \dots, -k_1) \\ &= (-1)^{\sum_i \{n_{r_i}^2 + n_{r_i} + 2\}/2} (-1)^{\sum_i n_{r_i} \sum_j n_{r_j}/2 - \sum_i n_{r_i}^2/2} V_{r_n \dots r_1}^{(n)}(-k_n, \dots, -k_1). \end{aligned} \quad (2.26)$$

Using the constraint on n_{r_i} given in (2.22), this can be rewritten as

$$\begin{aligned} V_{r_1 \dots r_n}^{(n)}(k_1, \dots, k_n)^* &= (-1)^{\sum_i n_{r_i}^2/2 + n} (-1)^{2n^2 - \sum_i n_{r_i}^2/2} V_{r_n \dots r_1}^{(n)}(-k_n, \dots, -k_1) \\ &= V_{r_n \dots r_1}^{(n)}(-k_n, \dots, -k_1). \end{aligned} \quad (2.27)$$

Reality of the action (2.19) follows immediately from this, (2.18), and the fact that under complex conjugation, a product of fields gets transformed to the product of the complex conjugate fields in the reverse order. For grassmann even fields this order reversal has no effect, but for grassmann odd fields this is related to the product in the original order by a sign.

It remains to prove (2.23) and (2.24). To prove (2.24) we note that after expressing Q_B and the states $\varphi_{r_1}(k_1)$ and $\varphi_{r_2}(k_2)$ in terms of the matter and ghost oscillators, the only explicit factors of i arise from the fact that in the expressions for the states φ_r 's we use the combination $-i\alpha_{-n}^\mu$ and $-i\bar{\alpha}_{-n}^\mu$. Now since the amplitude is Lorentz invariant, the α^μ 's must contract with each other in which case the factors of i combine in pairs to give a real number, or the $i\alpha_0^\mu$ factor acts on the vacuum producing a factor proportional to ik^μ . Since the latter factor remains invariant under the combined operation of complex conjugation and change of the sign of momenta, we get (2.24).

The proof of (2.23) can be given as follows. The general form of $\langle\langle \varphi_{r_1}(k_1) \cdots \varphi_{r_n}(k_n) \rangle\rangle$ is given by

$$\sum_{g=0}^{\infty} (g_s)^{2g} (2\pi i)^{-(3g-3+n)} \times \int_{\bar{\mathcal{R}}_{g,n}} dm^1 \wedge \cdots \wedge dm^{6g-6+2n} \left\langle b(v^{(1)}) \cdots b(v^{(6g-6+2n)}) \prod_{i=1}^n \varphi_{r_i}(k_i) \right\rangle_{\Sigma_{g,n}}. \quad (2.28)$$

First let us ignore the $(2\pi i)^{-(3g-3+n)}$ factor and the insertions of $b(v^{(i)})$'s in (2.28). In this case the correlation function

$$\left\langle \prod_{i=1}^n \varphi_{r_i}(k_i) \right\rangle_{\Sigma_{g,n}} \quad (2.29)$$

involves vertex operators constructed out of products of b , c , \bar{b} , \bar{c} , ∂X , $\bar{\partial} X$ and $e^{ik \cdot X}$ and their derivatives. Since the operator products of these operators have no explicit factor of i except for the factor of i accompanying each momentum factor k^μ , complex conjugation of the amplitude will have the effect of changing the sign of all the momentum factors, and mapping $\Sigma_{g,n}$ to $\Sigma_{g,n}^*$.² Therefore we get

$$\left(\left\langle \prod_{i=1}^n \varphi_{r_i}(k_i) \right\rangle_{\Sigma_{g,n}} \right)^* = \left\langle \prod_{i=1}^n \varphi_{r_i}(-k_i) \right\rangle_{\Sigma_{g,n}^*} \quad (2.30)$$

²One way to see this is to regard the genus g Riemann surface with n punctures as the result of plumbing fixture of several 3-punctured spheres. This allows us to express the correlation function on the genus g surface in terms of products of three point functions on the sphere. Let us for definiteness take the three insertion points on each sphere to be on the real axis, e.g. at 0, 1 and 2, and use the global coordinate on the complex plane in the plumbing fixture relations, e.g. for gluing the puncture at 0 on the i -th sphere to the puncture at 1 on the j -th sphere, use $z_i(z_j - 1) = q_{ij}$. Now since all the three point functions are real in the basis we have chosen — except for the factors of i multiplying k^μ — complex conjugation of the amplitude will have the effect of changing k^μ to $-k^\mu$, and complex conjugating all the variables $\{q_{ij}\}$ appearing in the plumbing fixture relations. The latter precisely takes us from $\Sigma_{g,n}$ to $\Sigma_{g,n}^*$.

Let us now study the effect of inserting the $b(v^{(\alpha)})$'s and the overall multiplicative factor of $(2\pi i)^{-(3g-3+n)}$ as given in (2.28). Complex conjugation of the multiplicative factor gives a factor of $(-1)^{3g-3+n}$. From (2.13) we see that we can represent the effect of the $b(v^{(\alpha)})$ insertions by changing the tensor product of external states $|\varphi_{r_1}(k_1)\rangle_{(1)} \otimes \cdots \otimes |\varphi_{r_n}(k_n)\rangle_{(n)}$ to

$$\prod_{\alpha=1}^{6g-6+2n} \left\{ \sum_{i=1}^n \sum_{m=-\infty}^{\infty} \left(v_{-m}^{(\alpha,i)} b_m^{(i)} + v_{-m}^{(\alpha,i)*} \bar{b}_m^{(i)} \right) \right\} |\varphi_{r_1}(k_1)\rangle_{(1)} \otimes \cdots \otimes |\varphi_{r_n}(k_n)\rangle_{(n)}. \quad (2.31)$$

The action of b_m and/or \bar{b}_m on the i -th state is to change the vertex operator to a different one that is still made of $b, c, \bar{b}, \bar{c}, \partial X, \bar{\partial} X, e^{ik \cdot X}$ and their derivatives without any explicit factor of i , except for the factors of i accompanying each factor of k_i^μ . Therefore the effect of complex conjugation of this amplitude will be to evaluate the correlation function on $\Sigma_{g,n}^*$, change the sign of all momenta, and replace $v_{-m}^{(j,i)}$ by its complex conjugate $(v_{-m}^{(j,i)})^*$. We shall now compare $(v_{-m}^{(j,i)})^*$'s with the $v_{-m}^{(j,i)}$'s associated with the tangent vectors of $\bar{\mathcal{R}}_{g,n}$ around the point $\Sigma_{g,n}^*$. There are several transformations involved in relating $v_{-m}^{(j,i)}$'s around the point $\Sigma_{g,n}^*$ to $v_{-m}^{(j,i)}$'s around the point $\Sigma_{g,n}$:

1. First of all since $\Sigma_{g,n}^*$ is obtained from $\Sigma_{g,n}$ by complex conjugating all transition functions, the $v_{-n}^{(j,i)}$'s will get complex conjugated:

$$v_{-n}^{(j,i)} \rightarrow v_{-n}^{(j,i)*}. \quad (2.32)$$

2. Since complex conjugation of the transition functions used in defining the Riemann surface induces complex conjugation of the coordinates on $\mathcal{M}_{g,n}$, the integration measure $\prod_i d\mu_i \wedge d\bar{\mu}_i$, where $\{\mu_i\}$ are the complex moduli, transforms to $\prod_i d\bar{\mu}_i \wedge d\mu_i = \prod_i (-d\mu_i \wedge d\bar{\mu}_i)$. Therefore the orientation of the moduli space picks a minus sign for each complex dimension. As a result, half of the $6g-6+2n$ $v_{-n}^{(j,i)}$'s also change sign besides being complex conjugated when we compare the tangent vectors of $\bar{\mathcal{R}}_{g,n}$ around $\Sigma_{g,n}$ and $\Sigma_{g,n}^*$. This gives a factor of

$$(-1)^{3g-3+n}, \quad (2.33)$$

when we compare the integration measure around $\Sigma_{g,n}$ with the integration measure around $\Sigma_{g,n}^*$.

These two effects together lead to the equation

$$\begin{aligned} & \left[(2\pi i)^{-(3g-3+n)} \int_{\bar{\mathcal{R}}_{g,n}} dm^1 \wedge \cdots \wedge dm^{6g-6+2n} \left\langle b(v^{(1)}) \cdots b(v^{(6g-6+2n)}) \prod_{i=1}^n \varphi_{r_i}(k_i) \right\rangle_{\Sigma_{g,n}} \right]^* \\ &= (2\pi i)^{-(3g-3+n)} \int_{\bar{\mathcal{R}}_{g,n}^*} dm^1 \wedge \cdots \wedge dm^{6g-6+2n} \left\langle b(v^{(1)}) \cdots b(v^{(6g-6+2n)}) \prod_{i=1}^n \varphi_{r_i}(-k_i) \right\rangle_{\Sigma_{g,n}}. \end{aligned} \quad (2.34)$$

Note that the $(-1)^{3g-3+n}$ given in (2.33) cancels the minus sign that arises from the complex conjugation of the $(2\pi i)^{-(3g-3+n)}$ factor. On the right hand side we have replaced the subscript $\Sigma_{g,n}^*$ of (2.30) by $\Sigma_{g,n}$ since conjugation operation is already encoded in the fact that the integration is performed over $\overline{\mathcal{R}}_{g,n}^*$. Using (2.34) and the fact that $\overline{\mathcal{R}}_{g,n} = \overline{\mathcal{R}}_{g,n}^*$, we recover (2.23).

This completes the proof of reality of the action of bosonic string field theory. One point worth mentioning here is that the reality conditions on all the fields are not unambiguously fixed by demanding the reality of the action. For example since for an m -point amplitude of states carrying ghost numbers n_1, \dots, n_m we have $\sum_k n_k = 2m$, the action remains invariant if we scale the fields as

$$\psi_r(k) \rightarrow e^{i\alpha(n_r-2)}\psi_r(k), \quad (2.35)$$

where α is an arbitrary real number. Therefore whatever reality condition was imposed on $\psi_r(k)$ can instead be imposed on $e^{i\alpha(n_r-2)}\psi_r(k)$ without affecting the reality of the action. Since this does not transform states in the physical sector (which have $n_r = 2$) this scaling has no effect on the physical S-matrix of the theory.

Once the reality condition on the string field is determined, we can use this to fix the overall sign of the action. Consider for example the string field component labelling the tachyon field $T(k)$

$$\int \frac{d^{26}k}{(2\pi)^{26}} T(k) \bar{c} c e^{ik \cdot X}. \quad (2.36)$$

According to the reality condition (2.16), $T(k)$ is the Fourier transform of a real scalar field. Using the normalization condition³

$$\langle k | c_{-1} \bar{c}_{-1} c_0 \bar{c}_0 c_1 \bar{c}_1 | k' \rangle = (2\pi)^{26} \delta^{(26)}(k + k'), \quad (2.37)$$

the kinetic term (2.8) includes a term

$$\frac{1}{4g_s^2} \int \frac{d^{26}k}{(2\pi)^{26}} \left(\frac{1}{2} k^2 - 2 \right) T(k) T(-k), \quad k^2 \equiv -(k^0)^2 + \vec{k}^2. \quad (2.38)$$

This is a wrong sign for the kinetic term. This can be repaired by the substitution $g_s^2 \rightarrow -g_s^2$. As long as this substitution is made in the interaction term as well, the action satisfies the requirement of gauge invariance, and we get a consistent string field theory. Furthermore, since the expansion of the action is in powers of g_s^2 , this does not introduce any extra factors of i , and the action remains real.

Before concluding this section we shall discuss the relation between the reality condition (2.16) and the one discussed in [2]. The reality condition in [2] was stated as the requirement that the hermitian conjugate and BPZ conjugate of a string field should have opposite signs. Therefore in order to translate this condition to a condition on the coefficients $\psi_r(k)$ in (2.7), we need to understand the difference between the action of hermitian conjugation and BPZ conjugation:

³This differs from that of [2] by a minus sign. The consequence of this different sign convention has been discussed in [6].

1. Hermitian conjugation replaces the ket state $|k\rangle$ by the bra $\langle -k|$, while BPZ conjugation replaces $|k\rangle$ by $\langle k|$.
2. Hermitian conjugation complex conjugates the coefficients $\psi_r(k)$ while BPZ conjugation leaves them unchanged.
3. Both hermitian conjugation and BPZ conjugation act in the same way on the oscillators $b_n, \bar{b}_n, i\alpha_n^\mu$ and $i\bar{\alpha}_n^\mu$, replacing n by $-n$ and also changing the signs of $i\alpha_n^\mu$ and $i\bar{\alpha}_n^\mu$. On the other hand hermitian conjugation takes c_n and \bar{c}_n to c_{-n} and \bar{c}_{-n} , while BPZ conjugation takes them to $-c_{-n}$ and $-\bar{c}_{-n}$, respectively. In arriving at these signs we have used the convention that BPZ conjugation involves conformal transformation of the vertex operator by the $SL(2, \mathbb{C})$ transformation $z \rightarrow 1/z$. This is to be contrasted with the BPZ transformation in open string theory where we use the $SL(2, \mathbb{R})$ transformation $z \rightarrow -1/z$.
4. Hermitian conjugation reverses the ordering of the oscillators as well as the relative position of the basis state $\varphi_r(k)$ and the coefficient $\psi_r(k)$, while BPZ conjugation leaves them unchanged.⁴

Therefore if a basis state $|\varphi_r\rangle$ has p_r number of b, \bar{b} oscillators and q_r number of c, \bar{c} oscillators, the reality condition of [2] may be stated as

$$\psi_r(k)^* = -(-1)^{q_r} (-1)^{(p_r+q_r)(p_r+q_r-1)/2} (-1)^{q_r-p_r} \psi_r(-k) \quad (2.39)$$

where the first minus sign is due to the requirement of relative minus sign between the hermitian and BPZ conjugation, the second factor is the effect of extra minus signs picked up by the c, \bar{c} oscillators, the third factor comes from having to rearrange the ghost oscillators, and the last factor comes from the reversal of the relative position of $\varphi_r(k)$ and $\psi_r(k)$. Using the relation $n_r = q_r - p_r$, one can see that this reduces to

$$\psi_r(k)^* = (-1)^{n_r(n_r+1)/2+1} (-1)^{n_r} \psi_r(-k). \quad (2.40)$$

This is not quite the same as (2.16), but differs from it by a factor of $(-1)^{n_r} = (-1)^{n_r-2}$. This difference however can be removed by redefining the reality condition on $\psi_r(k)$ by utilizing the freedom described in (2.35) with the choice $\alpha = \pi/2$.

3 Reality condition in heterotic and type II string field theory

In this section we shall determine the reality condition in the heterotic and type II string theories. We shall discuss the heterotic string theory in detail and then briefly mention the results for type II string theory.

The world-sheet theory of heterotic string in ten dimensions has additional fields besides what we have in the bosonic string theory. They include ten right moving fermions

⁴Since BPZ conjugation reverses the radial ordering, a more correct statement would be that BPZ conjugation also reverses the order of the operators, but for every reordering of a pair of grassmann odd operators, there is a minus sign.

ψ^μ , bosonic ghosts β , γ and an anti-chiral CFT of central charge 16, describing either $E_8 \times E_8$ or $SO(32)$ current algebra. The singular parts of the additional operator product expansions are:

$$\psi^\mu(z)\psi^\nu(w) = -\frac{1}{2(z-w)}\eta^{\mu\nu}, \quad \beta(z)\gamma(w) = \frac{1}{z-w} + \dots \quad (3.1)$$

The operator product of β and γ has non-standard sign convention, but this is the one that is compatible with the bosonization rules (3.4) and the operator product expansion (3.5) if we take into account the fact that ξ, η anti-commute with $e^{\pm\phi}$. Alternatively we could include an extra minus sign in the β - γ operator product expansion and include an extra minus sign in one of the terms in (3.4).

ψ^μ , β and γ have mode expansions:

$$\psi^\mu(z) = \sum_n \psi_n^\mu z^{-n-1/2}, \quad \beta(z) = \sum_n \beta_n z^{-n-3/2}, \quad \gamma(z) = \sum_n \gamma_n z^{-n+1/2}. \quad (3.2)$$

For the anti-chiral CFT of central charge 16, we shall not use any explicit representation, but denote by $|K\rangle$ a basis of Virasoro primary states satisfying

$$\langle K|L\rangle = \delta_{KL}, \quad \langle K|J(1)|L\rangle = \text{real}. \quad (3.3)$$

If we were representing the theory by a set of left-moving scalars Y^I then examples of such primary operators would have been ∂Y^I , $i \cos Y^I$, $i \sin Y^I$ etc. The full set of states in this CFT are obtained by acting on these primary states the Virasoro generators \bar{L}_n^G of this CFT. From now on we shall refer to this CFT as CFT_G , and the anti-holomorphic stress tensor of this CFT by \bar{T}^G .

For computing string amplitudes, we need to bosonize the β - γ system using the relations

$$\gamma = \eta e^\phi, \quad \beta = \partial \xi e^{-\phi}. \quad (3.4)$$

The leading terms in the operator product of ξ , η and $e^{q\phi}$ are

$$\xi(z)\eta(w) \simeq \frac{1}{z-w} + \dots, \quad e^{q\phi(z)}e^{q'\phi(w)} \simeq (z-w)^{-qq'}e^{(q+q')\phi(w)} + \dots \quad (3.5)$$

The fields ψ^μ , β , γ carry odd GSO parity whereas $e^{q\phi}$ carries GSO parity $(-1)^q$ for integer q . It follows from this that ξ and η have even GSO parity. We also assign $e^{q\phi}$ to have picture number q and ghost number 0, ξ to have picture number 1 and ghost number -1 and η to have picture number -1 and ghost number 1, so that β and γ have zero picture number, and ghost numbers -1 and 1 respectively.

The string field has two components: $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$. If we denote by \mathcal{H}_n the Hilbert space of GSO even states in string theory satisfying (2.5) and carrying picture number n , then $|\Psi\rangle$ takes value in $\mathcal{H}_{-1} \oplus \mathcal{H}_{-1/2}$ and $|\tilde{\Psi}\rangle$ takes value in $\mathcal{H}_{-1} \oplus \mathcal{H}_{-3/2}$. We shall denote by $\mathcal{H}_{\text{NS}} = \mathcal{H}_{-1}$ the Hilbert space of NS sector states and by $\mathcal{H}_R = \mathcal{H}_{-1/2} \oplus \mathcal{H}_{-3/2}$ the Hilbert space of R sector states. The string field theory action takes the form

$$S = g_s^{-2} \left[-\frac{1}{2} \langle \tilde{\Psi} | c_0^- Q_B \mathcal{G} | \tilde{\Psi} \rangle + \langle \tilde{\Psi} | c_0^- Q_B | \Psi \rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \{ \Psi^n \} \right], \quad (3.6)$$

where

$$\mathcal{G}|s\rangle = \begin{cases} |s\rangle & \text{if } |s\rangle \in \mathcal{H}_{\text{NS}} \\ \mathcal{X}_0 |s\rangle & \text{if } |s\rangle \in \mathcal{H}_R \end{cases}, \quad (3.7)$$

$$\mathcal{X}_0 \equiv \oint \frac{dz}{z} \mathcal{X}(z), \quad (3.8)$$

and $\mathcal{X}(z)$ is the picture changing operator (PCO) given by

$$\mathcal{X}(z) = \{Q_B, \xi(z)\} = c\partial\xi + e^\phi T_F - \frac{1}{4}\partial\eta e^{2\phi}b - \frac{1}{4}\partial\left(\eta e^{2\phi}b\right), \quad (3.9)$$

where

$$T_F(z) = -\psi_\mu \partial X^\mu. \quad (3.10)$$

The \oint in (3.8) includes the $1/2\pi i$ factor so that $\oint dz/z$ is normalized to 1. The definition of $\{\Psi^n\}$ is similar to that for the bosonic string theory with the following important differences. Now $\{A_1 \cdots A_m \hat{A}_1 \cdots \hat{A}_n\}$ for m NS-sector vertex operators $A_1, \cdots A_m$ and n R-sector vertex operators $\hat{A}_1, \cdots \hat{A}_n$ has, besides the insertion of the vertex operators and the b -ghost insertions, also insertion of PCO's. The locations of the PCO's appear as extra data in the definition of off-shell amplitudes, and so $\hat{\mathcal{P}}_{g,n}$ now has to be replaced by $\tilde{\mathcal{P}}_{g,m,n}$ whose base is the moduli space of ordinary Riemann surfaces with $m+n$ punctures together with the information on spin structure, and whose fiber, for a genus g amplitude, contains data on the choice of local coordinates around the punctures, as well as the locations of $2g - 2 + m + n/2$ PCO's. Given $\Sigma_{g,m,n} \in \tilde{\mathcal{P}}_{g,m,n}$, we define $\Sigma_{g,m,n}^* \in \tilde{\mathcal{P}}_{g,m,n}$ as the Riemann surface whose transition functions and local coordinates are complex conjugates of those of $\Sigma_{g,m,n}$, and for which the PCO locations are also complex conjugates of those on $\Sigma_{g,m,n}$. $\bar{\mathcal{R}}_{g,n}$ appearing in (2.10) now has to be replaced by $\bar{\mathcal{R}}_{g,m,n}$ — a (generalized) section of $\tilde{\mathcal{P}}_{g,m,n}$. Detailed procedure for choosing this section avoiding spurious poles can be found in [7]. We shall impose the additional restriction on $\bar{\mathcal{R}}_{g,m,n}$ that it is invariant under conjugation, i.e. if $\Sigma_{g,m,n} \in \bar{\mathcal{R}}_{g,m,n}$ then $\Sigma_{g,m,n}^* \in \bar{\mathcal{R}}_{g,m,n}$. A GSO even basis state and the string field component multiplying it are taken to be grassmann even for even ghost number states in the NS sector and odd ghost number states in the R-sector, and grassmann odd for odd ghost number states in the NS sector and even ghost number states in the R sector. For GSO odd basis states the grassmann parities are taken to be opposite. Even though the string field is always GSO even, the information on the grassmann parity of GSO odd states is sometimes useful during intermediate stages of manipulation, e.g. $e^{-\phi}$ will be taken to anti-commute with ψ^μ .

In the NS sector we construct the basis of states $|\varphi_r(k)\rangle$ by acting on the tensor product of the -1 picture vacuum $e^{-\phi}(0)e^{ik \cdot X}(0)|0\rangle$ with momentum k and some primary state $|K\rangle$ of CFT_G , by the oscillators of $b, c, \bar{b}, \bar{c}, \partial X, \bar{\partial} X, \beta, \gamma$ and \bar{T}^G , carrying a net GSO parity of -1 . We do not allow any extra factor of i in the definition of the basis states except for the factors of i accompanying the factors of k^μ . The vertex operators for these states can be built from linear combinations of GSO even products of (derivatives of) $\partial X^\mu, \bar{\partial} X^\mu, \psi^\mu, e^{ik \cdot X}, b, c, \bar{b}, \bar{c}, e^{q\phi}, \partial\phi, \partial\xi, \eta, \bar{T}^G$ and K , without any explicit factor of i . It follows

from the operator product expansions of the elementary fields, and (3.3), that the operator products of the φ_r 's, when expressed in terms of $\varphi_{r'}$'s, do not involve any factors of i , except for the factor of i accompanying each factor of k^μ .

Construction of the vertex operators in the Ramond sector also requires introduction of spin fields. The spin fields are of two types: chiral fields S_α and anti-chiral fields S^α . The mutually local GSO even combinations of spin fields in the matter and ghost sector are

$$e^{-(4n+1)\phi/2} S_\alpha, \quad e^{-(4n-1)\phi/2} S^\alpha, \quad (3.11)$$

and their derivatives and products with the NS sector GSO even operators. The operator products of these spin fields with each other and the GSO even NS sector vertex operators (e.g. $e^{-(2n+1)\phi}\psi^\mu$) can be computed from the following basic operator product expansions:

$$\begin{aligned} \psi^\mu(z) e^{-\phi/2} S_\alpha(w) &= \frac{i}{2} (z-w)^{-1/2} (\gamma^\mu)_{\alpha\beta} e^{-\phi/2} S^\beta(w) + \dots, \\ \psi^\mu(z) e^{-\phi/2} S^\alpha(w) &= \frac{i}{2} (z-w)^{-1/2} \gamma^{\mu\alpha\beta} e^{-\phi/2} S_\beta(w) + \dots, \\ e^{-\phi/2} S_\alpha(z) e^{-3\phi/2} S^\beta(w) &= \delta_\alpha^\beta (z-w)^{-2} e^{-2\phi}(w) + \dots, \end{aligned} \quad (3.12)$$

where γ^μ are ten dimensional γ -matrices, normalized as

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbf{1}, \quad (3.13)$$

where $(\gamma^\mu \gamma^\nu)_\alpha^\beta \equiv \gamma_{\alpha\delta}^\mu \gamma^{\nu\delta\beta}$ etc. We shall use a representation in which all the γ -matrices are purely imaginary and symmetric:⁵

$$(\gamma_{\alpha\beta}^\mu)^* = -\gamma_{\alpha\beta}^\mu, \quad (\gamma^{\mu\alpha\beta})^* = -\gamma^{\mu\alpha\beta}, \quad \gamma_{\alpha\beta}^\mu = \gamma_{\beta\alpha}^\mu, \quad \gamma^{\mu\alpha\beta} = \gamma^{\mu\beta\alpha}. \quad (3.14)$$

With this the right hand sides of (3.12) have real coefficients. If Γ^i for $1 \leq i \leq 8$ are the real 8×8 SO(8) gamma matrices satisfying $\Gamma^i (\Gamma^j)^T + \Gamma^j (\Gamma^i)^T = 2\delta^{ij}$ then a specific choice of SO(9,1) gamma matrices satisfying (3.14) is given by

$$\begin{aligned} \gamma_{\alpha\beta}^i &= \begin{pmatrix} 0 & i\Gamma^i \\ i(\Gamma^i)^T & 0 \end{pmatrix}_{\alpha\beta}, \quad \gamma^{i\alpha\beta} = \begin{pmatrix} 0 & -i\Gamma^i \\ -i(\Gamma^i)^T & 0 \end{pmatrix}_{\alpha\beta} \quad \text{for } 1 \leq i \leq 8, \\ \gamma_{\alpha\beta}^9 &= \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}_{\alpha\beta}, \quad \gamma^{9\alpha\beta} = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}_{\alpha\beta}, \quad \gamma_{\alpha\beta}^0 = \gamma^{0\alpha\beta} = \begin{pmatrix} iI & 0 \\ 0 & iI \end{pmatrix}_{\alpha\beta}. \end{aligned} \quad (3.15)$$

From these, and the grassmann parities of various operators described earlier, we can derive all other operator products, e.g. we have the following useful relations involving GSO

⁵Note that the overall phase of γ^μ can be changed by phase rotating S_α and S^α in the opposite direction without affecting the last equation in (3.12). Therefore the choice of γ^μ to be imaginary fixes the phases of S_α and S^α . The symmetry of γ^μ follows from the consistency of the operator product expansion (3.12). For example evaluation of the three point function $\langle e^{-\phi}\psi^\mu(z) e^{-\phi/2} S_\alpha(w) e^{-\phi/2} S_\beta(y) \rangle$ using (3.12) in different ways leads to the symmetry of $\gamma_{\alpha\beta}^\mu$.

even operators:

$$\begin{aligned}
 e^{-\phi}\psi^\mu(z)e^{-\phi/2}S_\alpha(w) &= \frac{i}{2}(z-w)^{-1}(\gamma^\mu)_{\alpha\beta}e^{-3\phi/2}S^\beta(w) + \dots, \\
 e^\phi\psi^\mu(z)e^{-3\phi/2}S^\alpha(w) &= -\frac{i}{2}(z-w)\gamma^{\mu\alpha\beta}e^{-\phi/2}S_\beta(w) + \dots, \\
 e^{-\phi/2}S_\alpha(z)e^{-\phi/2}S_\beta(w) &= -i(z-w)^{-1}\gamma_{\alpha\beta}^\mu e^{-\phi(w)}\psi_\mu(w) + \dots, \\
 e^{-\phi/2}S_\alpha(z)e^{\phi/2}S^\beta(w) &= -\delta_\alpha^\beta(z-w)^{-1} + \dots,
 \end{aligned} \tag{3.16}$$

etc. We shall choose the basis of $-1/2$ picture states $|\widehat{\varphi}_s(k)\rangle$ in the R-sector to be such that their vertex operators are constructed from products of (derivatives of) the operators appearing on the left hand side of the above equations, and other GSO even operators that were used to construct vertex operators for the basis states in the NS sector, without any explicit factor of i . A similar procedure is followed for the construction of the GSO even basis states $|\widetilde{\varphi}_s\rangle$ of $\mathcal{H}_{-3/2}$. In this case all the coefficients appearing in the operator product expansion of operators representing GSO even basis states in the NS and R sectors are manifestly real except for the factor of i multiplying each factor of k^μ .

During the evaluation of superstring amplitudes we also need insertion of PCO's given in (3.9). Again the operator product of these operators with each other and the NS and R sector vertex operators do not contain any explicit factors of i except those accompanying the k^μ factors.

To summarize, we have argued that as in the case of bosonic string theory, the operator product expansion of the vertex operators of basis states in the NS or R sectors, and the PCO's, do not contain any explicit factor of i , except that every factor of k^μ is accompanied by a factor of i . Using this one can argue, as in the case of bosonic string theory, that

$$\begin{aligned}
 \{\{\varphi_{r_1}(k_1) \cdots \varphi_{r_m}(k_m) \widehat{\varphi}_{s_1}(\ell_1) \cdots \widehat{\varphi}_{s_n}(\ell_n)\}\}^* \\
 = \{\{\varphi_{r_1}(-k_1) \cdots \varphi_{r_m}(-k_m) \widehat{\varphi}_{s_1}(-\ell_1) \cdots \widehat{\varphi}_{s_n}(-\ell_n)\}\},
 \end{aligned} \tag{3.17}$$

where φ_{r_i} 's denote basis of NS sector vertex operators of picture number -1 and $\widehat{\varphi}_{s_i}$'s denote basis of R-sector vertex operators of picture number $-1/2$. This assumes that we have chosen the integration slices $\overline{\mathcal{R}}_{g,m,n}$ of $\widetilde{\mathcal{P}}_{g,m,n}$ such that it is invariant under conjugation, including locations of the PCO's.

We can now begin discussing the reality of superstring field theory action. We begin by stating the reality condition on the field $|\Psi\rangle$. We expand the NS sector component of $|\Psi\rangle$ as

$$|\Psi_{\text{NS}}\rangle = \sum_r \int \frac{d^{10}k}{(2\pi)^{10}} \psi_r(k) |\varphi_r(k)\rangle \tag{3.18}$$

and the R component of $|\Psi\rangle$ as

$$|\Psi_R\rangle = \sum_s \int \frac{d^{10}k}{(2\pi)^{10}} \widehat{\psi}_s(k) |\widehat{\varphi}_s(k)\rangle. \tag{3.19}$$

We impose the reality condition

$$\psi_r(k)^* = (-1)^{n_r(n_r+1)/2+1} \psi_r(-k), \tag{3.20}$$

and

$$\widehat{\psi}_s(k)^* = -i(-1)^{(n_s+1)(n_s+2)/2}\widehat{\psi}_s(-k), \quad (3.21)$$

where n_r and n_s are ghost numbers of φ_r and $\widehat{\varphi}_s$ respectively. As will be seen in (3.27), the difference in the exponent of (-1) in (3.20) and (3.21) is due to the fact that in the R sector the grassmann parity of $\widehat{\varphi}_s$ is given by $(-1)^{n_s+1}$. Defining $\chi_r, \widehat{\chi}_s$ via⁶

$$\psi_r(k) = i^{n_r(n_r+1)/2+1}\chi_r(k), \quad \widehat{\psi}_s(k) = i^{(n_s+1)(n_s+2)/2+1/2}\widehat{\chi}_s(k), \quad (3.22)$$

the reality condition takes the form

$$\chi_r(k)^* = \chi_r(-k), \quad \widehat{\chi}_s(k)^* = \widehat{\chi}_s(-k). \quad (3.23)$$

As in the case of bosonic string theory, we could absorb the phase factors on the right hand sides of (3.22) into the definition of the basis states $|\varphi_r(k)\rangle$ and $|\widehat{\varphi}_s(k)\rangle$. In that case $\chi_r(k)$ and $\widehat{\chi}_s(k)$ will be directly interpreted as the coefficients of expansion of the string field in this basis. However we shall proceed with the original choice of basis.

Using the reality condition on $|\Psi\rangle$, we can proceed to check the reality of the interaction term involving $\{\Psi^n\}$. We write this as

$$\begin{aligned} \sum_n \frac{1}{n!} \{\Psi^n\} &= \sum_{m,n} \frac{1}{m!n!} \sum_{r_1, \dots, r_m} \sum_{s_1, \dots, s_n} (-1)^{\sum_{i<j} n_{r_i} n_{r_j} + \sum_{i<j} (n_{s_i}+1)(n_{s_j}+1) + \sum_{i,j} n_{r_i} (n_{s_j}+1)} \\ &\times \int \frac{d^{10}k_1}{(2\pi)^{10}} \cdots \frac{d^{10}k_m}{(2\pi)^{10}} \frac{d^{10}\ell_1}{(2\pi)^{10}} \cdots \frac{d^{10}\ell_n}{(2\pi)^{10}} \\ &\times V_{r_1 \dots r_m, s_1, \dots, s_n}^{(m,n)}(k_1, \dots, k_m, \ell_1, \dots, \ell_n) \chi_{r_1}(k_1) \cdots \chi_{r_m}(k_m) \widehat{\chi}_{s_1}(\ell_1) \cdots \widehat{\chi}_{s_n}(\ell_n), \end{aligned} \quad (3.24)$$

where

$$\begin{aligned} V_{r_1, \dots, r_m, s_1, \dots, s_n}^{(m,n)}(k_1, \dots, k_m, \ell_1, \dots, \ell_n) \\ = i^{\sum_{i=1}^m \{n_{r_i}(n_{r_i}+1)/2+1\} + \sum_{j=1}^n \{(n_{s_j}+1)(n_{s_j}+2)/2+1/2\}} \\ \times \{\varphi_{r_1}(k_1) \cdots \varphi_{r_m}(k_m) \widehat{\varphi}_{s_1}(\ell_1) \cdots \widehat{\varphi}_{s_n}(\ell_n)\}. \end{aligned} \quad (3.25)$$

The sign factors appearing in the first line of (3.24) arise from having to move the coefficients χ_r and $\widehat{\chi}_s$ through the operators φ_r and $\widehat{\varphi}_s$. We shall define the $V^{(m,n)}$'s for other ordering of the indices and arguments by appropriately rearranging the order of φ_r 's and $\widehat{\varphi}_s$'s inside $\{\}$ in (3.25) using the known grassmann parity of the basis states. Since the string field is always grassmann even, with this definition we also have

$$\begin{aligned} \sum_n \frac{1}{n!} \{\Psi^n\} &= \sum_{m,n} \frac{1}{m!n!} \sum_{r_1, \dots, r_m} \sum_{s_1, \dots, s_n} (-1)^{\sum_{i<j} n_{r_i} n_{r_j} + \sum_{i<j} (n_{s_i}+1)(n_{s_j}+1) + \sum_{i,j} n_{r_i} (n_{s_j}+1)} \\ &\times \int \frac{d^{10}k_1}{(2\pi)^{10}} \cdots \frac{d^{10}k_m}{(2\pi)^{10}} \frac{d^{10}\ell_1}{(2\pi)^{10}} \cdots \frac{d^{10}\ell_n}{(2\pi)^{10}} \\ &\times V_{s_n \dots s_1, r_m, \dots, r_1}^{(m,n)}(\ell_n, \dots, \ell_1, k_m, \dots, k_1) \widehat{\chi}_{s_n}(\ell_n) \cdots \widehat{\chi}_{s_1}(\ell_1) \chi_{r_m}(k_m) \cdots \chi_{r_1}(k_1). \end{aligned} \quad (3.26)$$

⁶We shall use the representation $i^{1/2} = \exp[i\pi/4]$.

$V^{(m,n)}$ has the symmetry properties

$$\begin{aligned} V_{\dots r_i r_j \dots}^{(m,n)}(\dots, k_i, k_j, \dots) &= (-1)^{n_{r_i} n_{r_j}} V_{\dots r_j r_i \dots}^{(m,n)}(\dots, k_j, k_i, \dots), \\ V_{\dots s_i s_j \dots}^{(m,n)}(\dots, \ell_i, \ell_j, \dots) &= (-1)^{(n_{s_i}+1)(n_{s_j}+1)} V_{\dots s_j s_i \dots}^{(m,n)}(\dots, \ell_j, \ell_i, \dots), \\ V_{\dots r_i s_j \dots}^{(m,n)}(\dots, k_i, \ell_j, \dots) &= (-1)^{n_{r_i}(n_{s_j}+1)} V_{\dots s_j r_i \dots}^{(m,n)}(\dots, \ell_j, k_i, \dots), \end{aligned} \quad (3.27)$$

where we have used that in the NS sector the grassmann parity is $(-1)^{n_r}$ whereas in the R sector the grassmann parity is $(-1)^{n_s+1}$. It follows from (3.17), (3.25), (3.27), and that the number n of Ramond sector states is always even, that

$$\begin{aligned} &\left(V_{r_1, \dots, r_m, s_1, \dots, s_n}^{(m,n)}(k_1, \dots, k_m, \ell_1, \dots, \ell_n) \right)^* \\ &= (-1)^{\sum_{i=1}^m \{n_{r_i}(n_{r_i}+1)/2+1\} + \sum_{j=1}^n \{(n_{s_j}+1)(n_{s_j}+2)/2+1/2\}} \\ &\quad \times V_{r_1, \dots, r_m, s_1, \dots, s_n}^{(m,n)}(-k_1, \dots, -k_m, -\ell_1, \dots, -\ell_n) \\ &= (-1)^{\sum_{i=1}^m \{n_{r_i}(n_{r_i}+1)/2+1\} + \sum_{j=1}^n \{(n_{s_j}+1)(n_{s_j}+2)/2+1/2\}} \\ &\quad \times (-1)^{\sum_{i<j} n_{r_i} n_{r_j} + \sum_{i<j} (n_{s_i}+1)(n_{s_j}+1) + \sum_{i,j} n_{r_i} n_{s_j} + 1} \\ &\quad \times V_{s_n, \dots, s_1, r_m, \dots, r_1}^{(m,n)}(-\ell_n, \dots, -\ell_1, -k_m, \dots, -k_1). \end{aligned} \quad (3.28)$$

Let us define

$$M = \sum_{i=1}^m n_{r_i}, \quad N = \sum_{j=1}^n n_{s_j}. \quad (3.29)$$

Ghost charge conservation gives

$$M + N = 2(m + n). \quad (3.30)$$

Using (3.29), (3.30), the exponents in the third and fourth lines of (3.28) may be written as, respectively,

$$\begin{aligned} &\sum_{i=1}^m \{n_{r_i}(n_{r_i}+1)/2+1\} + \sum_{j=1}^n \{(n_{s_j}+1)(n_{s_j}+2)/2+1/2\} \\ &= \frac{1}{2} \sum_{i=1}^m n_{r_i}^2 + \frac{1}{2} \sum_{j=1}^n (n_{s_j}+1)^2 + \frac{1}{2}M + m + \frac{1}{2}N + n \\ &= \frac{1}{2} \sum_{i=1}^m n_{r_i}^2 + \frac{1}{2} \sum_{j=1}^n (n_{s_j}+1)^2 + 2(m+n), \end{aligned} \quad (3.31)$$

and

$$\begin{aligned} &\sum_{i<j} n_{r_i} n_{r_j} + \sum_{i<j} (n_{s_i}+1)(n_{s_j}+1) + \sum_{i,j} n_{r_i} (n_{s_j}+1) \\ &= \frac{1}{2} \sum_{i,j} n_{r_i} n_{r_j} + \frac{1}{2} \sum_{i,j} (n_{s_i}+1)(n_{s_j}+1) \\ &\quad + \sum_{i,j} n_{r_i} (n_{s_j}+1) - \frac{1}{2} \sum_i n_{r_i}^2 - \frac{1}{2} \sum_j (n_{s_j}+1)^2 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2}M^2 + \frac{1}{2}(N+n)^2 + M(N+n) - \frac{1}{2}\sum_i n_{r_i}^2 - \frac{1}{2}\sum_j (n_{s_j} + 1)^2 \\
 &= \frac{1}{2}(M+N+n)^2 - \frac{1}{2}\sum_i n_{r_i}^2 - \frac{1}{2}\sum_j (n_{s_j} + 1)^2 \\
 &= \frac{1}{2}(2m+3n)^2 - \frac{1}{2}\sum_i n_{r_i}^2 - \frac{1}{2}\sum_j (n_{s_j} + 1)^2.
 \end{aligned} \tag{3.32}$$

Using (3.31), (3.32) and the fact that n is even, we can express (3.28) as

$$\left(V_{r_1, \dots, r_m, s_1, \dots, s_n}^{(m,n)}(k_1, \dots, k_m, \ell_1, \dots, \ell_n) \right)^* = V_{s_n, \dots, s_1, r_m, \dots, r_1}^{(m,n)}(-\ell_n, \dots, -\ell_1, -k_m, \dots, -k_1). \tag{3.33}$$

Substituting this into (3.24) and using (3.23) and (3.26) we see that this is exactly the relation needed for the reality of the interaction term of the string field theory action.

Let us now turn to the kinetic terms. For this we need to impose reality conditions on $|\tilde{\Psi}\rangle$ as well. We introduce basis states $|\tilde{\varphi}_r\rangle$ in $\mathcal{H}_{-3/2}$ following procedure similar to that in $\mathcal{H}_{-1/2}$, expand $|\tilde{\Psi}\rangle$ as

$$|\tilde{\Psi}\rangle = \sum_r \int \frac{d^{10}k}{(2\pi)^{10}} \xi_r(k) |\varphi_r(k)\rangle + \sum_s \int \frac{d^{10}k}{(2\pi)^{10}} \hat{\xi}_s(k) |\tilde{\varphi}_s(k)\rangle, \tag{3.34}$$

and impose the reality conditions

$$\xi_r(k)^* = (-1)^{n_r(n_r+1)/2+1} \xi_r(-k), \quad \hat{\xi}_s(k)^* = -i(-1)^{(n_s+1)(n_s+2)/2+1} \hat{\xi}_s(-k). \tag{3.35}$$

It is now easy to verify that each of the quadratic terms in the action satisfies the reality condition. Consider for example the term involving fields in $\mathcal{H}_{-3/2}$:

$$\frac{1}{2} \langle \tilde{\Psi} | c_0^- Q_B \mathcal{G} | \tilde{\Psi} \rangle = \frac{1}{2} \sum_{s_1, s_2} \int \frac{d^{10}k_1}{(2\pi)^{10}} \frac{d^{10}k_2}{(2\pi)^{10}} f_{s_1 s_2}(k_1, k_2) \hat{\xi}_{s_1}(k_1) \hat{\xi}_{s_2}(k_2) + \dots, \tag{3.36}$$

where

$$f_{s_1 s_2}(k_1, k_2) \equiv (-1)^{(n_{s_1}+1)(n_{s_2}+1)} \langle \tilde{\varphi}_{s_1}(k_1) | c_0^- Q_B \mathcal{G} | \tilde{\varphi}_{s_2}(k_2) \rangle. \tag{3.37}$$

It follows from (3.37), and the fact that the correlation functions of $\tilde{\varphi}_s$'s do not contain any explicit factor of i except those accompanying factors of k^μ , that

$$f_{s_1 s_2}(k_1, k_2)^* = f_{s_1 s_2}(-k_1, -k_2). \tag{3.38}$$

Therefore (3.36) gives, using (3.35)

$$\begin{aligned}
 \frac{1}{2} \langle \tilde{\Psi} | c_0^- Q_B \mathcal{G} | \tilde{\Psi} \rangle^* &= \frac{1}{2} \sum_{s_1, s_2} i^2 (-1)^{(n_{s_1}+1)(n_{s_1}+2)/2+1 + (n_{s_2}+1)(n_{s_2}+2)/2+1} \\
 &\quad \times \int \frac{d^{10}k_1}{(2\pi)^{10}} \frac{d^{10}k_2}{(2\pi)^{10}} f_{s_1 s_2}(-k_1, -k_2) \hat{\xi}_{s_2}(-k_2) \hat{\xi}_{s_1}(-k_1). \\
 &= \frac{1}{2} \sum_{s_1, s_2} i^2 (-1)^{(n_{s_1}+1)(n_{s_1}+2)/2+1 + (n_{s_2}+1)(n_{s_2}+2)/2+1} (-1)^{(n_{s_1}+1)(n_{s_2}+1)} \\
 &\quad \times \int \frac{d^{10}k_1}{(2\pi)^{10}} \frac{d^{10}k_2}{(2\pi)^{10}} f_{s_1 s_2}(-k_1, -k_2) \hat{\xi}_{s_1}(-k_1) \hat{\xi}_{s_2}(-k_2).
 \end{aligned} \tag{3.39}$$

Using the ghost charge conservation law $n_{s_2} = 4 - n_{s_1}$, it is easy to see that the net pre-factor is unity. Furthermore the signs of k_i in the arguments of $f_{s_1 s_2}$ and $\widehat{\xi}_{s_1}, \widehat{\xi}_{s_2}$ can be changed by variable redefinition. Hence we get

$$\frac{1}{2} \langle \widetilde{\Psi} | c_0^- Q_B \mathcal{G} | \widetilde{\Psi} \rangle^* = \frac{1}{2} \langle \widetilde{\Psi} | c_0^- Q_B \mathcal{G} | \widetilde{\Psi} \rangle. \quad (3.40)$$

Similar analysis can be used to establish the reality of all other quadratic terms in the action.

As in the case of bosonic string theory, the reality conditions (3.20), (3.21) and (3.35) are not fixed unambiguously. Besides the ambiguity described in (2.35) (with similar phase rotations acting on $\widehat{\psi}_s, \xi_r$ and $\widehat{\xi}_s$) we also have the freedom of multiplying each Ramond sector field by an additional factor of -1 under complex conjugation since the Ramond sector states always occur in pairs.

The analysis of the reality condition in type II string theory is similar. There are now four sectors. The action takes the same form as given in (3.6) with $|\Psi\rangle$ taking value in $\mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1,-1/2} \oplus \mathcal{H}_{-1/2,-1} \oplus \mathcal{H}_{-1/2,-1/2}$, and $|\widetilde{\Psi}\rangle$ taking value in $\mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1,-3/2} \oplus \mathcal{H}_{-3/2,-1} \oplus \mathcal{H}_{-3/2,-3/2}$. The definition of $\{\}$ now includes insertion of holomorphic and anti-holomorphic PCO's, and the operator \mathcal{G} takes the form

$$\mathcal{G}|s\rangle = \begin{cases} |s\rangle & \text{if } |s\rangle \in \mathcal{H}_{\text{NSNS}} \\ \mathcal{X}_0 |s\rangle & \text{if } |s\rangle \in \mathcal{H}_{\text{NSR}} \\ \bar{\mathcal{X}}_0 |s\rangle & \text{if } |s\rangle \in \mathcal{H}_{\text{RNS}} \\ \mathcal{X}_0 \bar{\mathcal{X}}_0 |s\rangle & \text{if } |s\rangle \in \mathcal{H}_{\text{RR}}. \end{cases} \quad (3.41)$$

Analysis similar to the one for heterotic string field theory can be carried out here. It is easy to see that the reality condition on the fields is determined simply by whether the field is fermionic or bosonic, since this determines the relation between the grassmann parity and the ghost number. Therefore once we have chosen a basis generated by products of derivatives of $e^{ik \cdot X}$ and standard ghost and matter fields with real operator product expansion coefficients, and expanded the string field in such a basis, the reality condition on the string field in the NSNS sector and RR sector takes the form given in (3.20) whereas for string fields in the RNS or NSR sectors, the reality condition takes the form of (3.21).

Again, once the reality condition is determined, we can use it to fix the sign of the action. For the heterotic string theory we can consider the component of the string field $|\Psi\rangle$ describing a graviton field component $h_{12}(k)$:

$$\int \frac{d^{10}k}{(2\pi)^{10}} h_{12}(k) \bar{c} c e^{-\phi} \psi^1 \bar{\partial} X^2 e^{ik \cdot X}. \quad (3.42)$$

In the NS sector we can take $\widetilde{\Psi} = \Psi$. Substitution into (3.6) yields the kinetic term

$$- \frac{1}{32g_s^2} \int \frac{d^{10}k}{(2\pi)^{10}} [k^2 h_{12}(k) h_{12}(-k) + \text{terms proportional to } k^1, k^2]. \quad (3.43)$$

This has the correct sign and no $g_s^2 \rightarrow -g_s^2$ substitution is necessary.

In type II string theory we work in the NSNS sector where again we can set the string field components $\tilde{\Psi}$ and Ψ to be equal from the beginning. We can again consider the string field component describing the graviton field

$$\int \frac{d^{10}k}{(2\pi)^{10}} h_{12}(k) \bar{c} c e^{-\phi} \psi^1 e^{-\bar{\phi}} \bar{\psi}^2 e^{ik \cdot X}. \quad (3.44)$$

Substitution into the action (3.6) gives

$$\frac{1}{32g_s^2} \int \frac{d^{10}k}{(2\pi)^{10}} [k^2 h_{12}(k) h_{12}(-k) + \text{terms proportional to } k^1, k^2]. \quad (3.45)$$

This has wrong sign and hence we need to make a $g_s^2 \rightarrow -g_s^2$ substitution to get the correct sign of the kinetic term.

4 Reality condition as a relation between hermitian conjugation and BPZ conjugation

We have seen that in the case of bosonic string theory, the reality condition can be interpreted as the equality between hermitian conjugate and BPZ conjugate of the string field up to a sign. We shall now show that the same result holds for superstring theory provided we choose the hermitian conjugation rules of various fields appropriately, and exploit the ambiguities mentioned in the paragraph below (3.40) judiciously.⁷ We shall discuss the case of heterotic string theory in detail; the analysis for type II string theory is very similar and will be mentioned briefly at the end.

We begin by defining the action of hermitian conjugation on various oscillators. We choose the following definitions

$$\begin{aligned} (i\alpha_n^\mu)^\dagger &= -i\alpha_{-n}^\mu, & (i\bar{\alpha}_n^\mu)^\dagger &= -i\bar{\alpha}_{-n}^\mu, & b_n^\dagger &= b_{-n}, & c_n^\dagger &= c_{-n}, & \bar{b}_n^\dagger &= \bar{b}_{-n}, & \bar{c}_n^\dagger &= \bar{c}_{-n}, \\ (\bar{L}_n^G)^\dagger &= \bar{L}_{-n}^G, & (\psi_n^\mu)^\dagger &= \psi_{-n}^\mu, & \beta_n^\dagger &= \beta_{-n}, & \gamma_n^\dagger &= -\gamma_{-n}. \end{aligned} \quad (4.1)$$

It is easy to verify that the hermitian conjugation rules given above preserve the (anti-)commutation relations between the oscillators. Besides this we shall assume that for integer q hermitian conjugation takes $e^{q\phi}$ to $e^{q\bar{\phi}}$, and it takes the vacuum $|k, K\rangle \equiv e^{ik \cdot X}(0)|0\rangle \otimes |K\rangle$ to $\langle -k, K|$ where $\langle k, K|$ denotes the BPZ conjugate of $|k, K\rangle$. Finally it reverses the order of all the operators and complex conjugates any multiplicative coefficient. Action of BPZ conjugation is standard, except that due to half-integral dimensions carried by various operators we have to choose the phase appropriately. For example acting on a primary operator $V(z, \bar{z})$ of dimension (\bar{h}, h) the BPZ conjugation gives a multiplicative factor of $(-1/z^2)^h (-1/\bar{z}^2)^{\bar{h}} V(1/z, 1/\bar{z})$, and we have to fix the phase for non-integer values of $(h - \bar{h})$. We use the convention that acting on a primary operator V of dimension (\bar{h}, h) at $z = \bar{z} = 1$, the BPZ conjugation takes it to

$$e^{-i\pi(h-\bar{h})} V(1). \quad (4.2)$$

⁷I would like to thank Barton Zwiebach for prompting me to investigate this.

We also use the convention of [5] to define star conjugation as the hermitian conjugation followed by inverse of BPZ conjugation. Our goal will be to check if the string field, satisfying the reality conditions (3.20), (3.21), has simple properties under star conjugation.

We begin our analysis with the NS sector. Let us consider an arbitrary basis state obtained by acting on the vacuum $e^{-\phi}(0)|k, K\rangle$ by various modes of $b, \bar{b}, c, \bar{c}, \psi^\mu, \beta, \gamma, i\alpha^\mu, i\bar{\alpha}^\mu$ and \bar{L}^G without any additional factor of i . Let n_b be the number of b, \bar{b} oscillators, n_c be the number of c, \bar{c} oscillators, n_ψ be the number of ψ^μ oscillators, n_β be the number of β oscillators and n_γ be the number of γ oscillators. We also define

$$n_{bc} = n_c - n_b, \quad n_{\beta\gamma} = n_\gamma - n_\beta, \quad n = n_{\beta\gamma} + n_{bc}. \quad (4.3)$$

n is the total ghost number of the state. Without loss of generality, we can arrange the oscillators such that all the b, c, \bar{b}, \bar{c} oscillators are to the extreme left, all the β, γ oscillators are grouped together in the middle and all the ψ^μ oscillators are to the extreme right, sitting next to the vacuum $e^{-\phi}(0)|k, K\rangle$. Locations of the $i\alpha_{-n}^\mu, i\bar{\alpha}_{-n}^\mu$ and \bar{L}_{-n}^G oscillators will not matter; we can for definiteness fix them to be at the left of the ψ_{-n}^μ 's.

We shall now collect various factors that arise from star conjugation. First of all star conjugation changes the relative position of the b, c, \bar{b}, \bar{c} oscillators with respect to the combination of ψ^μ oscillators and $e^{-\phi}$. This gives a factor

$$(-1)^{(n_b+n_c)(n_\psi+1)}. \quad (4.4)$$

Star conjugation of the β, γ system gives a factor

$$e^{i\pi(3n_\beta-n_\gamma)/2}(-1)^{n_\gamma} = e^{-3i\pi n_{\beta\gamma}/2}. \quad (4.5)$$

Here the first factor on the left hand side is the inverse of the phase described in (4.2) picked up during inverse BPZ conjugation while the second factor is due to the minus sign picked up by the γ oscillators during hermitian conjugation. The star conjugation of b, c, \bar{b}, \bar{c} oscillators gives

$$(-1)^{n_c}(-1)^{(n_b+n_c)(n_b+n_c-1)/2} = (-1)^{n_{bc}(n_{bc}+1)/2}. \quad (4.6)$$

The first factor again comes from (4.2) while the second factor is due to the reversal of order of the b, c, \bar{b}, \bar{c} oscillators due to hermitian conjugation. The star conjugation of ψ^μ and $e^{-\phi}$ system generates the factor

$$(-1)^{n_\psi(n_\psi+1)/2}e^{i\pi(n_\psi+1)/2}. \quad (4.7)$$

The first factor is due to the reversal of the order of the operators due to hermitian conjugation and the second factor comes from (4.2). There are no factors from star conjugation of the $i\alpha_{-n}^\mu, i\bar{\alpha}_{-n}^\mu$ or \bar{L}_{-n}^G since hermitian and BPZ conjugation act on them in the same way, and they are all grassmann even.

Now the condition that the state is GSO even requires $n_\psi + 1 - n_{\beta\gamma}$ to be even. Hence we write

$$n_\psi + 1 - n_{\beta\gamma} = 2m, \quad m \in \mathbb{Z}. \quad (4.8)$$

Using (4.3), (4.8) we can express the product of (4.4)–(4.7) as

$$(-1)^{n(n+1)/2}. \quad (4.9)$$

This is the sign picked up by a basis state under star conjugation. The only other change is the replacement of k^μ by $-k^\mu$. Combining this with (3.20) we see that the phase picked up by ψ_r combines with that of φ_r to give a net factor of -1 . However since star conjugation exchanges the positions of ψ_r and φ_r , it produces another factor of $(-1)^{n_r-2}$ since this is the grassmann parity of ψ_r and φ_r . This factor, however, can be removed by modifying the reality condition on ψ_r using the freedom described in (2.35) with the choice $\alpha = \pi/2$. With this, the reality condition on the NS sector string field may be written as the statement that the star conjugation changes the sign of the string field.

The above analysis can be extended to the Ramond sector with a few changes. We represent the basis states in the same way, as oscillators acting on the Ramond vacuum state $e^{-\phi/2}S_\alpha(0)|k, K\rangle$. We define the action of hermitian conjugation on the operator $e^{-\phi/2}S_\alpha$ such that it differs from BPZ conjugation by a factor of i . This may seem unusual, but is needed for example to satisfy

$$\langle b|\mathcal{O}|a\rangle^* = \langle a_{hc}|\mathcal{O}^\dagger|b_{hc}\rangle \quad (4.10)$$

with the choice $a = e^{-\phi/2}S_\alpha$, $b = e^{-\phi/2}S_\beta$ and $\mathcal{O} = e^{-\phi}\psi^\mu$. Here the subscript hc denotes hermitian conjugation. Using the hermitian conjugation rules for $|a\rangle$ and $|b\rangle$ defined above, the result that $\mathcal{O}^\dagger = -\mathcal{O}$ due to the exchange of $e^{-\phi}$ and ψ^μ induced by hermitian conjugation, and the fact that γ^μ 's are imaginary and symmetric, we get both the left and the right hand sides to be $i\gamma_{\alpha\beta}^\mu/2$. However without the factor of i included in the definition of the hermitian conjugation of $e^{-\phi/2}S_\alpha$, the two sides will differ by a minus sign.

We can now compare the signs picked up by a general basis state under star conjugation with the corresponding analysis in the NS sector. The first difference is the extra factor of i in the hermitian conjugation of the Ramond vacuum. The second difference arises from the fact that the phase (4.2) picked up by the operator $e^{-\phi/2}S_\alpha$ during BPZ conjugation has already been taken into account in the statement that star conjugation of this gives a factor of i ; so we do not need to include this in the analog of (4.7). However like $e^{-\phi}$, the new operator is also grassmann odd, hence the effect of reordering generates the same factor as in (4.7). This has the effect of changing (4.7) to

$$(-1)^{n_\psi(n_\psi+1)/2}e^{i\pi n_\psi/2}. \quad (4.11)$$

Another change occurs in (4.8) since the requirement of GSO even state now requires $n_\psi - n_{\beta\gamma}$ to be even. Hence we write

$$n_\psi - n_{\beta\gamma} = 2m, \quad m \in \mathbb{Z}. \quad (4.12)$$

Using (4.3), (4.12) we can now express the product of (4.4)–(4.6), (4.11) and i as

$$-i(-1)^{(n+1)(n+2)/2}. \quad (4.13)$$

This is the sign picked up by $\widehat{\varphi}_s$ under star conjugation. Combining this with (3.21), and the fact that the exchange of the position of $\widehat{\psi}_s$ and $\widehat{\varphi}_s$ under star conjugation gives rise to an additional factor of $(-1)^{n_s-1}$, we see that the reality condition on the string field requires that the Ramond sector string field picks up a factor of $-(-1)^{n_s-1}$ under star conjugation. However we can remove the last $(-1)^{n_s-1}$ factor by a combination of the freedom described in (2.35) with $\alpha = \pi/2$ and the freedom of multiplying each R sector states by an additional factor of -1 under star conjugation. With this change of star conjugation rules of the R sector field, we see that the reality condition on the R sector fields can be stated as the condition that they change sign under star conjugation.

The analysis for the $\widetilde{\Psi}$ field is similar, with $e^{-3\phi/2}S^\alpha$ replacing $e^{-\phi/2}S_\alpha$ as the operator creating the Ramond vacuum state.

Let us now briefly discuss the analysis in the type II string theory. We begin with the NSNS sector. In this case the vacuum is obtained as $e^{-\phi}(0)e^{-\bar{\phi}}(0)|k\rangle$ and we have additional oscillators of $\bar{\psi}^\mu$. Now if we go back to the analysis of NS sector of the heterotic string theory, we can see that there was no difference in our treatment of ψ^μ and $e^{-\phi}$ since they have identical transformation under star conjugation and carry identical grassmann and GSO parity, and due to this all relations (4.4)–(4.8) involved only the combination $n_\psi + 1$. So we can now repeat the analysis by grouping the oscillators of ψ^μ , $\bar{\psi}^\mu$ and the $e^{-\phi}$, $e^{-\bar{\phi}}$ together. If n_ψ is the total number of ψ^μ and $\bar{\psi}^\mu$ oscillators then the analysis of the NS sector for heterotic string theory can be repeated without any change, except that all factors of $(n_\psi + 1)$ will be replaced by $n_\psi + 2$ to take into account the presence of the $e^{-\bar{\phi}}$ factor.⁸ Since the reality conditions on the string field components take form identical to that for the NS sector of the heterotic string theory, we conclude that the string field satisfying the reality condition changes sign under star conjugation.

The analysis in the RR sector follows in similar fashion once we note that the operator $e^{-\phi/2}S_\alpha e^{-\bar{\phi}/2}\bar{S}_\beta$ that creates the RR vacuum is invariant under star conjugation. The two factors of i picked up by the two spin fields cancel against the minus sign that comes from having to exchange their positions. Therefore the analysis of the heterotic string NS sector can now be repeated with the replacement of $n_\psi + 1$ by n_ψ since we no longer have the $e^{-\phi}$ factor. The result again is the change in sign of the string field under star conjugation.

For the RNS and NSR sectors, we can use the analysis used for the R sector of the heterotic string theory with the replacement of n_ψ by $n_\psi + 1$ to take into account the extra factor of $e^{-\phi}$ or $e^{-\bar{\phi}}$ coming from the NS sector on the right or left. Therefore the phase picked up by the basis states under star conjugation is identical to that for the R sector of heterotic string theory. Since the component fields also pick up the same phases as in the R sector of the heterotic theory, we again conclude that the string field changes sign under star conjugation.

⁸There is actually an additional factor of -1 for every half-integer weight anti-holomorphic field since, according to (4.2), under BPZ conjugation a half integer weight anti-holomorphic field picks an additional -1 factor compared to a holomorphic field of the same weight. However since GSO projection ensures that the total number of half-integer weight anti-holomorphic fields for any state is even, this does not introduce any net factor.

5 Non-trivial background

Let us now consider the effect of putting string theory in a non-trivial space-time background. First let us consider the case of bosonic string theory. In this case in the world-sheet theory certain number of X^μ 's will be replaced by an internal CFT of the same central charge. As long as we can choose a basis of conformal primary operators of this CFT that has the property that the three point functions on the sphere of all the primary operators are real for real insertion points, we can build the basis in \mathcal{H} by taking the tensor product of descendants of these basis states and the basis states in the CFT involving the remaining X^μ 's and the ghost fields, constructed in the manner described in section 2. With this choice of basis the reality of the string field theory action follows in a manner identical to that in the flat background.

Note that this choice of basis states will typically make the basis non-eigenstates of the charge operators. For example for a compact internal dimension Y , it will require us to use the operators $e^{iK \cdot Y} + e^{-iK \cdot Y}$ and $-i(e^{iK \cdot Y} - e^{-iK \cdot Y})$ as basis states, instead of $e^{\pm iK \cdot Y}$. However the proof of the reality of the string field theory is simplest in this basis.

We also need to ensure that the kinetic terms of the string fields, obtained after imposing the reality condition, come with the correct choice of sign. This will require the primary states of the basis, chosen in the manner described above, to have positive BPZ inner product, with BPZ inner product as defined in (2.9). In order that a CFT provides a consistent background for formulating string theory, its correlation functions must satisfy these conditions.

The analysis for heterotic and type II superstring theories are similar. For example for the heterotic string theory we have to assume that the internal superconformal field theory has a basis of GSO odd and GSO even primary states in the NS sector, and GSO odd and GSO even primary states in the Ramond sector such that the 3-point functions of $e^{-2n\phi}$ multiplied by GSO even states in the NS sector, $e^{-(2n+1)\phi}$ multiplied by the GSO odd states in the NS sector, $e^{-(4n+1)\phi/2}$ multiplied by the GSO even states in the R sector and $e^{-(4n-1)\phi/2}$ multiplied by the GSO odd states in the R-sector are all real for real insertion points. Once this condition is satisfied, the reality of the string field theory action follows from the same line of argument as in the case of string theory in flat space-time background. We also need to check that once the reality condition is satisfied, the kinetic terms have the correct sign. The requirement of reality of the type II string field theory action is a straightforward generalization of these constraints.

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